THE FORTY-EIGHTH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE forty-eighth annual meeting of the American Association for the Advancement of Science was held in Columbus, Ohio, August 21–26. There were 352 members and associates in attendance and 273 papers were presented before the sections. The members came from twenty-eight different states, the District of Columbia, and Canada. A considerable number of members of affiliated societies were also in attendance, many of whom took part in the proceedings of the sections in which they were interested. Their presence increased the general scientific interest; and in view of the desire of the various sections to have still closer relations with the respective affiliated societies, authority was given to the officers of the Association whereby any section may arrange for a joint meeting with societies of a similar scope. It is hoped that the AMERICAN MATHEMATICAL SOCIETY and the Society for the Promotion of Engineering Education may meet next year with Sections A and D respectively. Probably several other societies will hold joint meetings with the corresponding sections of the Association.

The next meeting of the Association will be held in New York City during the last week of June, 1900, under the presidency of Professor R. S. Woodward, Columbia University. Professor Asaph Hall, Jr., Michigan University, will be vice-president of Section A, and Dr. W. M. Strong, Yale University, will be secretary. It is hoped that the change in the time of these meetings from the latter part of August to the last week in June will tend to increase the attendance. This change seemed especially desirable for next year since it will enable many to attend the meetings of the Association on their way to the Paris exposition.

The meetings of the section of mathematics and astronomy were well attended. The officers of this section were: vice-president, Alexander Macfarlane; secretary, J. F. Hayford; councilor, C. L. Doolittle; sectional committee, Alexander Macfarlane, J. F. Hayford, G. B. Halsted, R. D. Bohannan, W. M. Strong. The Council elected the following mathematicians and astronomers to fellowship in the Association: S. M. Barton, W. P. Durfee, O. L. Fassig, R. A. Fessenden, C. E. Furness, R. A. Harris, G. A. Hill,

J. E. Keeler, W. H. Metzler, G. W. Meyers, D. A. Murray, W. M. Strong, J. H. Tanner, A. G. Webster.

The opening address of the Vice-President of Section A was on "The fundamental principles of algebra." It has been published in *Science*, September 15, 1899. The following papers were presented at the meetings of this section :

(1) Professor G. B. HALSTED : "Report on progress in non-euclidean geometry."

(2) Professor L. E. DICKSON : "Report on progress in the theory of linear groups."

(3) Professor ASAPH HALL, Jr.: "The aberration constant from observations of Polaris."

(4) Professor R. W. MCFARLANE : "Ancient eclipses and chronology."

(5) Professor H. C. LORD : "Some points in the design of a spectroscope."

(6) Professor J. V. COLLINS : "Note on Grassmann's proof that there can be but two kinds of lineal multiplication of two factors."

(7) Professor G. J. STOKES : "The theory of mathematical inference."

(8) Mr. J. W. DAVIS : "Internal forces that generate stellar atmospheres."

(9) Professor R. A. FESSENDEN : "The determination of the nature of electricity and magnetism, including a determination of the density of the ether."

(10) Dr. G. A. MILLER : "On the commutators of a group."

(11) Dr. S. KIMURA : "Linear vector functions."

Three of the titles that appeared on the programme are not reproduced here on account of the fact that these papers did not reach the secretary in time to be presented at the meeting. In the absence of the authors the papers by Professor Stokes and Dr. Kimura were presented by Professors Macfarlane and Shaw respectively. The paper by Professor Fessenden was read before a joint session of Sections A and B and will be reviewed in connection with the papers of Section B. Professor Dickson's report has already appeared in the October number of the BULLETIN. Dr. Miller's paper will appear in a later number. Abstracts of the other papers are given below.

Professor Halsted contrasted the neglect of Lobachévski and Bolyai with the present interest in non-euclidean geometry as exhibited by the appearance in the *Monist* of a long article on this subject by Poincaré. The investigations

of Sophus Lie and the contributions of Gérard were discussed. Burnside's articles in the *Proceedings of the London Mathematical Society* were interpreted. The new book, Universal algebra, by Whitehead, and the important discoveries freeing us from the misleading conceptions of curvature of space in non-euclidean geometry were elucidated. The philosophic bearings of the new space ideas were illustrated with special reference to B. A. W. Russell's book. Killing and Clifford spaces were noted. The reproduction of the non-euclidean classics, especially by Engel and Stäckel, was described, also the work of the Hungarian academy of sciences. The history of the subject (Loria) and finally the bibliography closed the report, which has recently appeared in *Science*, October 20, 1899.

Professor Hall stated that observations of the zenith distances of Polaris for the determination of the aberration constant have been made with the meridian circle of the Detroit Observatory since about May 1, 1898. The direct observations made during the interval from May, 1898 to July, 1899, have been roughly reduced and give the values 20.″60 and 20.″58, from pointings above and below the pole respectively. These seem rather large, the correct value being supposed to be about 20.″50. By observing above and below the pole the latitude variation is eliminated, but the effect of temperature on the instrumental readings needs to be followed very carefully. It is to be hoped that this work can be kept up through the year 1900. It has been suggested by several astronomers and physicists that a correct theoretical treatment of aberration has not yet been obtained. The values of the aberration constant recently found vary a great deal, ranging from about 20.″31 to 20.″66. An increase in the value of this constant would indicate, of course, a decrease in the usually adopted value of the solar parallax.

The burden of the argument in Professor McFarlane's paper was that eclipses have not aided in fixing any historical event in its proper place in history; that the difficulty is not mathematical, but arises from want of exactness in historical records; and that this want of exactness arose from the want of a fixed time from which to reckon, because the writers of ancient histories, so-called, gave, in general, no dates at all. Moreover, all references to eclipses are vague in the extreme. Taking the most celebrated of all the cases—the eclipse of Thales—it was

shown that Sir George Airy and Mr. Baily differed about 25 years in the date of the eclipse, and that other astronomers, and celebrated scholars, in England, France, Germany, and elsewhere, have set the time in nine different years scattered over a period of more than forty-six years. If the most celebrated case thus falls, there is no standing for the others.

At one place in the demonstration given on page 28 of the *Ausdehnungslehre* of 1862, an equation can be derived from the preceding in a somewhat different manner than that given. As this theorem is fundamental in Grassman's system, Dr. Collins worked out the following alternative proof:

In the derivation of equation (*c*) from

$$(b) \qquad \Sigma x_{r,u}x_{s,v}(a_{r,s}[e_ue_v] + a_{s,r}[e_ve_u]) = 0$$

by successive substitution of $2x_{a,c}$ and $x_{a,c}$ instead of $+1$ and -1 as values of a particular $x_{r,u}$, we get

$$\Sigma x_{s,v}(a_{a,s}[e_ce_v] + a_{s,a}[e_ve_c]) + 3x_{a,c}a_{a,a}[e_ae_a] = 0,$$

in which the summation does not extend to $s = a$, $v = c$ at the same time. Inserting the missing term,

$$\Sigma x_{s,v}(a_{a,s}[e_ce_v] + a_{s,a}[e_ve_c]) + 2x_{a,c}a_{a,a}[e_ae_a] = 0.$$

By putting a particular $x_{s,v}$, as $x_{b,d}$ first equal to 2 and then equal to 1, we have finally

$$(c) \qquad a_{a,b}[e_ce_d] + a_{b,a}[e_de_c] = 0.$$

The derivation, by symmetry, of equation (*b*) from the equation which precedes it

$$\Sigma x_{r,u}x_{s,v}a_{r,s}[e_ue_v] = 0$$

is very interesting on account of its simplicity, but suggests logical doubts. Mr. Collins points out that the terms of

$$\Sigma x_{r,u}x_{s,v}a_{r,s}[e_ue_v]$$

which have the same coefficients (products of x's) are of two types, viz.,

$$x_{r,u}x_{s,v}(a_{r,s}[e_ue_v] + a_{s,r}[e_ve_u]), \qquad x_{r,u}x_{r,u}a_{r,r}[e_ue_u],$$

whose combination, in view of

$$\Sigma x_{r,u}x_{s,v}a_{r,s}[e_ue_v] = 0,$$

gives (*b*) at once. Thus the direct derivation gives the same result as symmetry.

The paper of Professor Stokes, who occupies the chair of logic in Queens College, Cork, compared the fundamental equations $x^2 = x$ and $x\,(1-x) = 0$ of Boole's calculus with the fundamental equations $l_r^2 = 0$ and $l_r l_s = -l_s l_r$ of Grassmann's calculus of extension, and stated :—"Whereas the first set of equations belong to an algebra of self-identical unrelated units, the second set belong to an algebra in which relations, synthesis, reference beyond self, is essential."

In a paper on the "Imaginary of logic" before the British Association in 1898 the author put forward the view that as the square root of a positive quantity is + or −, the square root of a negative quantity may be expected to be + and −, in view of the logical relation between *and* and *or* pointed out originally by DeMorgan, and subsequently by Schroeder. He considers that this view receives support from the paper of Kempe on the theory of mathematical form. He concludes as follows:—"the fertile proportions of mathematics from which its wealth of content and the treasures of mathematical knowledge are drawn, are not synthetical in the sense in which Kant and the empiricists alike maintain them to be, viz : that the truths preëxist and are then seen to be synthetical, the synthetical character being as it were something subsequent to the content of the proportions and attached to it as it were adjectively ; but in this sense that those proportions are themselves the product of pure synthesis, that the very possibility of advance from entity to entity, unit to unit, or relation to correlate, determines all those laws which mathematics is employed in exploring and tracing into all their consequences, and which are infinitely more fruitful than the analytical laws of formal logic or the calculus of classes and statements. Pure synthesis generally is that 'necessity of matter' of which Hamilton spoke; the principle of material consequences which characterizes every genuine department of mathematics and defies further logical analysis."

The conclusion reached in the paper by Mr. Davis is that every celestial body wholly gaseous, large or small, composed of a mixture in which the proportion of heavier gases is considerable, necessarily generates an outrushing atmosphere, and that this atmosphere ceases to exist only upon the exhaustion of the lighter gases or the liquefaction of the central core. The atmosphere is exceedingly tenuous.

In case of the sun the yearly shrinkage of the nucleus is known to be only a few hundred feet. The consequent extrusion of the lighter gases annually adds to the atmosphere a few thousand feet, and this in the course of the year is spread throughout the solar system.

Dr. Kimura's paper is a development of many formulæ for the application of the nonion operator φ and its differential nabla $\nabla\!\!\!/$ to mathematical physics. Expressions were given for the moduli of the characteristic equations of powers of φ; expressions for powers of φ in terms of any three other powers of φ; expressions for the reduction of products such as $\varphi^{r}\varphi^{s}$ or $\varphi^{s}\varphi^{r}$, where φ' is the *conjugate* of φ; expressions involving the self-conjugate part of φ, and the rotation-vector of φ; and finally a considerable study of the operator

$$\nabla\!\!\!/ = \begin{vmatrix} \frac{\partial}{\partial a_1} & \frac{\partial}{\partial b_1} & \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial a_2} & \frac{\partial}{\partial b_2} & \frac{\partial}{\partial c_2} \\ \frac{\partial}{\partial a_3} & \frac{\partial}{\partial b_3} & \frac{\partial}{\partial c_3} \end{vmatrix} \varphi, \quad \text{where } \varphi = \begin{pmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{pmatrix}$$

It is proper to add that the formulæ in the main are applicable only to linear vector operators which have: (*a*) distinct roots. (*b*) two equal roots, but reducible to the form

$$\varphi = -\frac{g_1\rho_1 s\rho_2\rho_3() - g_1\rho_2 s\rho_3\rho_1() - g_2\rho_3 s\rho_1\rho_2()}{s_1\rho_1\rho_2\rho_3}.$$

(*c*) three equal roots, but reducible to the form $\varphi = g$.

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THE THEORY OF FUNCTIONS.

Introduction to the Theory of Analytic Functions. By J. HARKNESS and F. MORLEY. London, Macmillan and Co., 1898. 8vo, xv + 336 pp.

THE authors, already well known through their Treatise on the theory of functions, have laid the mathematical public under a new obligation by their "Introduction to the theory of analytic functions." The name which the authors have chosen for their new book gives only an inadequate idea of its scope and object; on the one hand it is not an introduction in the sense of a *first* introduction to the theory of functions; on the other hand it is much more than an introduction; it may be shortly described as a very complete *treatise on Weierstrass's general theory of functions* with applications to elliptic and algebraic functions (Chapters VI., XIX., XXI.), preceded by an introduction devoted to the number concept and the geometric interpretation of complex quantities (Chapters I.–V.), and followed by a short account of some of the leading ideas of Riemann and Cauchy (Chapters XX., XXII.).

I. *The Introductory and Concluding Chapters.*

The book begins with a sketch of the system of real numbers considered as *ordinal numbers* (Chapter I.). A row of objects is considered with regard to their order, say from left to right. To count the objects means to label them, not primarily to say how many there are; and the integers are, primarily, mere labels. The object after which we begin to count, is labelled 0; if there are also objects to the left of it, they are labelled $-1, -2$, etc. Fractions are introduced by a process of relabelling: Pick out of the original row the objects $p, 2p, 3p \cdots$, and relabel them $1, 2, 3, \cdots$; if μ is not divisible by p, the object originally labelled μ is relabelled $\frac{\mu}{p}$. At the same time rules are given to decide whether two rational numbers are equal, and if not, which is the greater. A different kind of relabelling, viz., a change of origin, is used to define addition without reference to quantity. Also special irrationals like $\sqrt{2}$ can be introduced by a similar process of relabelling; but for the general definition of irrationals Dedekind's idea of a cut in

the totality of rational numbers is adopted and it is shown how every decimal, finite or infinite, defines a cut.

Chapter II. gives the *geometric representation of complex numbers* and of the elementary algebraic operations with them, with numerous applications to geometrical problems. Along with the usual representation of a complex number by a point or by a stroke, the interpretation as an operator is emphasized, and, as in quaternions, the consideration of the quotient precedes that of the product.

Chapter III. contains a detailed study of the *bilinear transformation* interpreted as a one-to-one correspondence between two planes. The starting point is the geometric interpretation of the two equations

$$am\left(\frac{x-x_1}{x-x_2}\right)=\alpha \text{ and } \left|\frac{x-x_1}{x-x_2}\right|=\rho\,;$$

the former represents an arc of a circle through x_1, x_2, the latter a complete circle "about x_1, x_2," that is a circle with respect to which x_1, x_2 are inverse points; the two circles are orthogonal.

By this means a very elegant treatment of the general bilinear transformation is obtained; and at the same time a simple geometric interpretation of two harmonic pairs.

Further the anharmonic ratio of four points and the isogonality of the transformation are considered and finally the subject is connected with the theory of absolute inversion in space. An interesting—apparently new—theorem is given, viz., that *every bilinear transformation between two planes is equivalent to two inversions in space.*

In Chapter IV. "*Geometric theory of the logarithm and exponential*" the authors intend to give an elementary geometric treatment of $\log \xi$, analogous to the way in which $\sin \xi$ is treated in trigonometry. For this purpose, the existence of the equiangular spiral is assumed, "which may be constructed by placing a right cone with axis vertical and attaching a thread AP to a point P of it. The point P is held in the horizontal plane through the vertex O; the thread is then wound round the cone, without being allowed to slip. The curve described by the point P in its plane is the spiral in question, and it can be proved in an elementary way that it cuts all rays from O at a constant angle. Choose this angle to be $\pi/4$; and then measuring the angle θ from the point where $OP=\rho$ has the value 1, define

$$\theta=\log\rho."$$

It follows then easily that

$$D \log \rho = \frac{1}{\rho}$$

$$\log \frac{\rho_1}{\rho} = \log \rho_1 - \log \rho.$$

The logarithm of a complex quantity x_1 is defined by

$$\int_1^{x_1} \frac{dx}{x},$$

and the mapping by means of $\log x$ is discussed with application to Mercator's projection.

The exponential is defined as the inverse of the logarithm. The mapping by means of $x = \exp y$ leads in a natural way to the "Napierian motion" defined by $x_t = x_0 \exp bt$, which plays an important part in the following Chapter V., in which the *bilinear transformation* is taken up again and interpreted as a *transformation of a plane into itself*, produced by a motion of all the points of the plane.

The same Chapter V. deals also with periodic transformations, in particular those of period 2 and 3. I call special attention to the elegant construction of the double points of an involution which is given by two of its pairs.

These are some of the principal features of the introductory chapters. The presentation is elegant, suggestive, and original, excellently adapted for an intelligent student who has already gone through a first introductory course in the theory of functions, such as given, for instance, in Durège, or Burkhardt, or in the function theoretical chapters of Chrystal.

But these chapters would be open to serious objections if they should be intended—as some passages in the Preface seem to indicate—for a beginner, that is a student who is introduced for the first time to the theory of functions. To such a student the many new ideas and new methods of reasoning of the theory of functions, together with the mystery still surrounding the conception of imaginaries, with which he has made only an occasional and mostly very unsatisfactory acquaintance in algebra and calculus, all these things combined usually offer considerable difficulties, and a book intended for the use of beginners ought to make the entrance as easy as possible.

A beginner will hardly be benefited by the delicate and half-philosophical developments of Chapter I. on ordinal numbers; the more so, as they do not give a complete ac-

count of the matter. In Chapter II. a more systematic, even dry and school-like treatment would better meet his needs, and, for a beginner, the elegant treatment of the bilinear transformation in Chapters III. and V. should be preceded by a more elementary chapter preparing by a study of the simplest special cases: $x + b$, ax, $1/x$ for the most general case.

In Chapter IV. the highly original construction of $\log \xi$ by means of the cone and the thread will puzzle most readers, unless a further explanation be added of what is meant by "winding the thread round the cone without allowing it to slip." But there is a more serious objection to this chapter: Its proofs have only a preliminary character and lay no claim to rigor; and yet some of its results are presupposed in Chapter XIII., where a rigorous analytical theory of the exponential and logarithm is given; so in particular in § 96 where the tacit assumption is made that the function $\exp x$ defined in § 95 is identical with the function $\exp x$ defined in § 32.

We now turn to the *concluding chapters:*

Chapter XX. illustrates the construction of *Riemann surfaces* for algebraic functions by a number of simple examples, which furnish at the same time an opportunity for the distinction between simply and multiply connected surfaces. One paragraph gives an elementary explanation of the fundamental concepts of the theory of groups of linear transformations.

The final Chapter XXII. is devoted to *Cauchy's definition of a monogenic function.* The necessary and sufficient conditions for the existence of a derivative of a function $u + iv$ of $x + iy$ are given, one of the necessary conditions (which had been overlooked until very recently) being that u and v must have total differentials. A special paragraph gives an excellent critical discussion of the difficulties underlying Cauchy's definition. One of these is of particular interest: the question is raised whether the continuity of the derivative is a necessary consequence of its existence, a question which has in the meantime been settled by Goursat's discovery that Cauchy's theorem can be proved without presupposing the continuity of the derivative.

In connection with the conditions for monogeneity, the potential is introduced, the equipotential problem is explained and solved for some simple cases, and the solution applied to the problem of the conform representation of a straight line upon a polygon. Finally Green's theorem is given and applied to Cauchy's theorem.

The authors say in the preface: "We had to make a choice between the methods of Cauchy and those of Weierstrass." I must confess that I cannot see the necessity of such exclusiveness. On the contrary, I believe the authors could materially increase the usefulness of their book by working over, in a new edition, the Chapters II.–V., XX., XXII., into an introductory part, adapted in the first place to the needs of the beginner, giving an elementary exposition of Cauchy's theory with emphasis upon conform representation, accepting to a certain extent intuitional proofs. Such an introduction would be the best preparation for Weierstrass's theory; it would culminate in Cauchy's theorem on Taylor's series, and here the point of junction with Weierstrass would be gained. A critical chapter showing the hidden difficulties of Cauchy's theory and the insufficiency of geometrical proofs would be an excellent link and would prepare for the standpoint of absolute rigor to be adhered to afterwards. For a student will only properly appreciate the stern beauty of Weierstrass's theory, if it has come to him as a salvation out of the doubts and difficulties of the geometrical methods.

II. *Weierstrass's Theory.*

The rest of the book gives a systematic exposition of Weierstrass's theory of functions, excellent in its general arrangement, clear and rigorous in the details, with the exception of some minor flaws,* too few in number and too insignificant to mar the general impression of excellency. We make the following subdivisions:

A. The *Preparatory Chapters* (VI.–IX.).

B. The *General Theory of Analytic Functions* (X.–XII., XIV., XVI.).

* For the benefit of the readers of the "Introduction" I call attention to the following corrigenda and omissions:

(1) P. 91, the quantity γ must be assumed to be some value *actually taken by* $|f(x)|$.

(2) P. 173, the condition for the edges should be

$$\xi = +\frac{\pi}{2},\ \eta \gtreqless 0\,;\quad \xi = -\frac{\pi}{2},\ \eta \gtreqless 0.$$

(3) P. 179, replace γ by "the maximum value of $|f(x)|$ in the closed region $(R' + D - \delta_1)$."

(4) P. 253, line 4, $\frac{w}{2}$ must not be a period.

(5) P. 270, line 6, $4\omega_\mu$, $4\omega_\nu$ are not primitive.

(6) P. 295, line 14, a_n is an integral function of $a_1, a_2, \cdots, a_{n-1}$ and of the c_{ik}'s *with real positive coefficients.*

C. Applications to *Special Classes of Functions* (XIII., XV., XVII.–XIX., XXI.).

A. The Preparatory Chapters.

They lay the foundation of the theory by an account of the general theory of limits and infinite series and a chapter on rational functions.

In Chapter VI. the principal theorems concerning *limits* of infinite sequences and *continuous functions* of one or two real variables are given on the basis of Dedekind's definition of irrationals; the theorem on uniform continuity leads to a discussion of the general problem of uniform convergence to a limit.

The following chapter (VII.) develops the fundamental properties of *rational functions of a complex variable* and gives the definitions of "limit, continuity, derivative, zero, pole," for the special case of these functions, but in such a manner that they can at once be generalized.

Chapter VIII. treats of the *convergence of infinite series with complex terms.* Simple tests of convergence are given and the question of association of terms, the distinction between absolutely and conditionally convergent series, and the conversion of a single series into a double series and *vice versa*, are carefully discussed.

Next follows (Chapter IX.) the subject of uniform convergence of real series, its connection with continuity and an application to real power series, including their behaviour in the frontier points.

Two desiderata must be mentioned with respect to these preparatory chapters: (1) An exhaustive theory of irrationals should be given, and (2) a scientific theory of complex numbers from the standpoint of linear associative algebra. They are two of the cornerstones of Weierstrass's theory and should not be omitted in an account of it otherwise so detailed and complete.

B. The General Theory of Analytic Functions.

The reader is now thoroughly prepared for the general theory of analytic functions which is developed in the following chapters along the line of Weierstrass's lectures, still with many minor deviations.

Chapter X. treats of the fundamental properties of *power-series*, their domain of convergence, the uniformity of their convergence and the criteria of identity of two power series.

The next Chapter XI. on *operations with power series* is

based upon Weierstrass's theorem on series of power series. From this theorem (for which in many cases Cauchy's theorem on double sums may be substituted) follow easily the rules for the multiplication and division of power series, Taylor's theorem for power series along with the rules for differentiation of power series and series of power series.

The *continuation of a power series* (Chapter XII.) is only another aspect of Taylor's theorem, and thus we have reached Weierstrass's definition of an analytic function, viz. a power series together with the totality of its continuations, direct or indirect. Two theorems are necessary to complete the definition: (1) A power series $P(x - x_0)$ and any one of its direct continuations $P(x - x_0 | x_1)$ coincide throughout their common domain of convergence, (2) The series $P(x - x_0)$ is always a direct or indirect continuation of $P(x - x_0 | x_1)$. From the latter theorem follows that, no matter what element of an analytic function we take as starting point, we always obtain the same totality of power series.

Instead of this second theorem, Harkness and Morley use for the same purpose what they call "*standard chains*" for the continuation of a power series from a point a to a point b, that is, a series of points.

$$a, x_1, x_2, \cdots, x_n, b$$

such that each point is in the domain of the *following* as well as the preceding point. The same series of points may then be used for the continuation from b to a. But it should be stated and proved that if $P(x - b)$ is a direct or indirect continuation of $P(x - a)$ reached by a non-reversible chain, the same continuation can also be reached by a reversible ("standard") chain.

Frequently it is convenient to think of the *continuation* of a power series as taking place *along a continuous path* from a to b. The path must be such that it is possible to choose on it a series of points $x_1, x_2, \cdots, x_n$ between a and b such that each point x_{i+1} together with the arc $x_i x_{i+1}$ of the curve is contained in the domain of the point x_i, and it can be proved* that the value reached in b depends only on the path and not on the choice of the intermediate points. Thus a one valued function is defined along the path.

The authors use such paths of continuation later on to prove the theorem that an analytic function is monodromic in a simply connected region which contains no singular point.

* See C. Jordan, Cours d' analyse, 2d ed., I., no. 343.

But these geometrical considerations are rather out of harmony with their strictly Weierstrassian surroundings; to make them rigorous it would be necessary to give purely arithmetical definitions of a "continuous path," "circuit," "simply connected region."

The general theory is completed in Chapter XIV., which is devoted to the *singular points*. The existence of at least one singular point on the circumference of the circle of convergence of a power series is proved and the characteristic difference between essential and non-essential singular points of one valued functions discussed and it is shown how certain classes of one valued functions are characterized by the nature of their singularities: the rational integral, the rational fractional, the transcendental integral, and the transcendental fractional functions.

It is to be regretted that the authors have helped further to increase the lamentable confusion which already exists in our *terminology* by the introduction of a new term for the concept which occurs most frequently in the theory of functions, viz., the aggregate of properties of a function, necessary and sufficient for its expansion into a power series. It seemed for a while that the term *holomorphic*, introduced by Briot and Bouquet, had gained general recognition; it was adopted by C. Jordan (1st ed.), by Forsyth, and by Harkness and Morley (Treatise). And now the latest three text-books on the theory of functions abandon it again: Jordan (2d ed.) returns to Cauchy's original "*synectic*," Burkhardt uses the term "*regular*," occasionally used by Weierstrass, and Harkness and Morley introduce the term "*analytic about a point*," which moreover has the disadvantage that the word analytic is already in use—aside from its general meaning—in a different technical sense in Weierstrass's theory.

Chapter XVI. represents a new departure from the ordinary lines of Weierstrass's theory; it develops *Cauchy's theorems on definite integrals on the basis of Weierstrass's concept of an analytic function*. For this purpose it is first shown* that the definite integral—considered as a limit of a sum—

* A difficulty of uniform convergence is overlooked in the proof: It is tacitly assumed that for every ε a δ exists such that

$$\left| \frac{(x_\lambda + h_\lambda)^{n+1} - x_\lambda^{n+1}}{h_\lambda} - (n+1)x_\lambda^n \right| < \varepsilon$$

as soon as all $|h_\lambda| < \delta$, *no matter where* x_λ *is taken on the path*; that is, uniform differentiability of x^n along the path is assumed. It could easily be proved that this condition is always satisfied.

of the function x^n, taken from x_0 to x_1 along any path L of finite length, has the value

$$\frac{x_1^{n+1} - x_0^{n+1}}{n+1}.$$

Next it is proved that for the integration of a power series $P(x) = \Sigma a_n x^n$ integration by terms is allowed, provided the path lies wholly within the circle of convergence. For such a path we have therefore the result

$$\int_{x_0}^{x_1} (\Sigma a_n x^n)\, dx = \Sigma a_n \frac{x_1^{n+1} - x_0^{n+1}}{n+1} = \bar{P}(x_1) - \bar{P}(x_0),$$

if we denote by $\bar{P}(x)$ the indefinite integral of the series $P(x)$. But this difference is Weierstrass's definition of a definite integral for a power series, and thus the two definitions are reconciled.

To extend the result to the case where the path of integration is not wholly contained in the domain of one element, but still a "path of continuation" for the analytic function under consideration, interpolate on the path a finite number of points chosen, as explained above, and apply the previous theorem to each pair of successive points. The element $P(x)$ continued along L furnishes, along L, a one valued function $f(x)$; the indefinite integral $\bar{P}(x)$ can be continued along the same path and furnishes another one valued function $F(x)$, and all along L: $\dfrac{dF(x)}{dx} = f(x)$. The final result is then

$$\int_L f(x)dx = F(x_1) - F(x_0).$$

From this point of view, Cauchy's theorem dwindles down almost to a triviality: it is self-evident if the region around which we integrate is contained in the domain of an element of the integrand, and in the general case it is an immediate consequence of the theorem of Article 107. But we must remember that Cauchy's original theorem states much more than in the present connection, since it does not presuppose the expansibility of the function into a power series.

Cauchy's integral and the theorems on residues follow now easily, and a collection of interesting applications to the computation of real definite integrals concludes the chapter.

C. Special Classes of Functions.

The principles of the preceding general theory are illustrated by a series of applications to special classes of functions, which occupy the remaining chapters of the book.

Chapter XIII. gives an analytic theory of the *Exponential and Logarithm.* The exponential is defined by the series, and the logarithm as its inverse. Special attention is paid to the "chief values" of the logarithm and the allied many valued functions; they are distinguished by capital initials from the general values, and defined as follows:

$$-\pi < Am\, x \leqq \pi$$

$$\operatorname{Log} x = \operatorname{Log} |x| + i Am x.$$

The trigonometric functions $\cos x$, $\sin x$ are defined by the series and their properties are derived from those of the exponential. The corresponding conform representation problems are studied, and utilized for the investigation of the chief values of the inverse function.

In Chapter XV., after a succinct account of the theory of infinite products based upon the theorem that

$$\lim_{n=\infty} \exp x_1 \cdot \exp x_2 \cdots \exp x_n = \exp\left(\lim_{n=\infty} (x_1 + x_2 + x_n)\right),$$

Weierstrass's *construction of the most general transcendental integral function with given zeros* is developed, with applications to the trigonometric functions and to the Γ-functions.

In Chapter XVII. *Laurent's theorem* is proved by means of Cauchy's integral, and applied to a discussion of the isolated singularities of one valued functions, to the expansion of a periodic function into a Fourier series, and to a succinct and elegant treatment of the *ϑ-functions:* If $g(x)$ denotes the infinite product

$$g(x) = (1-x)(1-rx)(1-r^2x)\cdots \qquad (|r|<1),$$

then

$$f(x) = g(x)\, g\left(\frac{r}{x}\right)$$

is a one valued analytic function with the zeros

$$\cdots, \frac{1}{r^2}, \frac{1}{r}, 1, r, r^2, \cdots$$

and the only singular points $x = 0$ and $x = \infty$. It can be expanded into a Laurent series, and the substitution $r = q^2$, $-x = q \exp 2\pi i v$ transforms $f(x)$ into the fundamental ϑ-function, and the principal properties of the ϑ-functions

flow easily from the double representation of $f(x)$ by an infinite product and by an infinite series.

The following two chapters (XVIII., XIX.) develop *the theory of elliptic functions*, on the whole along the lines of Weierstrass's lectures but with many deviations and simplifications in the detail arrangements. Thus instead of Weierstrass's half-period $\omega_2 = \omega_1 + \omega_3$, its negative is used so that

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

This change which, it seems, has been simultaneously and independently introduced by Harkness and Morley,* by Study, and by C. Jordan, is an important improvement, since it leads to a complete symmetry with respect to a cyclic permutation of the indices 1, 2, 3.

Considerable simplifications are obtained by an extensive use of integration round a period parallelogram; I mention in particular the proof of the theorem that for every elliptic function $f(u)$ the sum of the zeros is congruent to the sum of the poles, by means of the integral

$$\int \frac{uf'(u)}{f(u)}\, du$$

taken round the period parallelogram.

In Chapter XXI. a sketch of the elementary properties of algebraic functions from the standpoint of Weierstrass is given. It is proved that an irreducible algebraic equation $F(x, y) = 0$ defines a single analytic function y of x; its singular points are determined, the Puiseux expansions in the vicinity of ordinary branch points are derived and the problems of multiple points of the curve $F = 0$ is solved for the simplest case of ordinary double points.——

The presentation is throughout clear, and numerous and well selected examples and exercises illustrate the general theorems.

The authors have adopted, for their exposition of the theory of functions, Weierstrass's standard of absolute arithmetical rigor. They insist that the intuitional method is not in itself sufficient for the superstructure of the theory; and systematically exclude it from all *proofs*. At the same time a happy conciseness of expression prevents the clumsiness which so frequently accompanies rigorous deductions.

* Treatise (1893), p. 301; Study, Sphaerische Trigonometrie and Elliptische Functionen (1893), p. 190; C. Jordan, Cours d'Analyse (1894), II., p. 337.

A wise moderation has been exercised in the selection of topics, and the arrangement of the rich material is pedagogical as well as logical. The concept of the analytic function takes the central place, everything else is subservient to it, either by preparing for it or by illustrating it. By this means, a harmony and equilibrium between the different parts is attained, which impart to the whole in a remarkable degree the character of an organic unity.

Thus the authors have succeeded in producing not only a work of high scientific and pedagogical value but at the same time of a singular beauty and elegance. But there are numerous beauties of detail as well, for which, however, the reader must be referred to the book itself. A certain freshness and originality pervade the whole, even in places where the authors follow along beaten tracks, and give at every turn evidence of the complete mastery of the subject with which the book is written.

OSKAR BOLZA.

UNIVERSITY OF CHICAGO,
June 30, 1899.

McAULAY'S OCTONIONS.

Octonions. By A. McAULAY, M.A. Cambridge, The University Press, 1898. 8vo, 253 pp.

THIS treatise is a development of Clifford's biquaternions with applications to ordinary space. The starting point of the analysis is quaternions, combined later with methods from Grassman's Ausdehnungslehre.

The development is open to criticism, as a work for beginners, because of its extremely refined formal character. This is perhaps unavoidable because the book was first compiled as a scientific paper, and, as the author says, he did not feel justified in recasting the whole appropriately. Discussions of formal laws and expert reasoning upon terms that are imperfectly defined, relying upon subsequent developments to bring out their full meaning, are not conducive to clearness of apprehension on the part of a learner. Perhaps as good a review, therefore, as can be made of the book, which is indeed an extensive and thorough development of what must prove to be a valuable analysis, is to give a brief and clear exposition of the octonion system.

The octonion is a quantity defined by four numbers called its *tensor*, *scalar*, *convert*, and *pitch*, and an axis having a fixed position in space. This makes in all eight numbers, upon

which the octonion depends, and hence its name. Let q, r be two quaternions, and O a given point. Then q, r, O determine the octonion as follows: the four numbers, as named above, are Tq, Sq, Sr, and $S \cdot r/q$; the axis is a line parallel to Vq, whose vector distance from O is $V \cdot Vr/Vq$. McAulay expresses this octonion by $Q = q + r\Omega$, where Ω is a unit of separation; but it is better, because different points may be employed in the definition of the same octonion, to exhibit the point of reference, say $Q = qO + rO\Omega$; and we shall presently see that we may write $O\Omega = \Omega$.

If instead of O, we employ another point P, at a vector distance ρ from O, certain changes must be made in r to give the same octonion; q evidently will remain unchanged since the numbers Tq, Sq, and the direction of the axis (parallel to Vq), completely determine q. Since Sr remains unchanged, r can change only by a vector σ, and since $S \cdot r/q$ remains unchanged, therefore $S \cdot \sigma/q = 0$, or σ is a vector perpendicular to Vq. Finally, σ/Vq = increase of perpendicular distance of the axis = component of PO perpendicular to $Vq = -V\rho Vq \cdot /Vq$, and hence $\sigma = -V\rho Vq$. Thus,

$$Q = qO + rO\Omega = qP + (r - V\rho Vq)P\Omega.$$

When $Vq = 0$, the axis is either at infinity ($Vr \neq 0$) or indeterminate ($Vr = 0$); in either case the quaternion components of Q are independent of the reference point, so that it may be omitted. In particular, $rO\Omega = rP\Omega = r\Omega$ say; so that in effect $O\Omega = P\Omega = \Omega$.

Addition is defined by adding coefficients of corresponding units, *viz.*, if

$$Q = qO + r\Omega = qP + (r - V\rho Vq)\Omega,\ Q' = q'O + r'\Omega = \text{etc.},$$

then, $$Q + Q' = (q + q')O + (r + r')\Omega = (q + q')P + [r + r' - V\rho V(q + q')]\Omega.$$

Here we add Q and Q' as to the reference point O, and also, as to the reference point P, and find the two sums to be equal. Addition is thus seen to be independent of the reference point O or P, and this means, of course, that the sum is determined wholly by the numbers and axis that define each term of the sum. Addition is also, from quaternion laws, associative and commutative.

Multiplication is defined by making it distributive and the units O, Ω commutative with everything, with the table $O^2 = O$, $\Omega^2 = 0$, and $O\Omega = \Omega O = \Omega$ as found above. It follows that multiplication is also independent of the reference point, viz.:

$$QQ' = qq'O + (qr' + rq')\Omega = qq'P + (qr' + rq' - V\rho Vqq')\Omega,$$
$$= qq'P + (qr' + rq' - qV\rho Vq' - V\rho Vq \cdot q')\Omega,$$

since $$V \cdot \rho Vqq' = qV\rho Vq' + V\rho Vq \cdot q'.$$

Multiplication is also associative, but not in general commutative. An octonion scalar $x + y\Omega$ is commutative with everything; nonaxised octonions are commutative with each other; a nonaxised and an axised octonion are commutative only when the (secondary) direction (Vr) of the first is parallel to the axis of the second; axised octonions are commutative only when their axes coincide.

Any octonion may be resolved into four commutative factors in one and only one way. viz.:

A *tensor* Tq, an *additor* $1 + a\Omega$, a *versor* $Uq \cdot A$, and a *translator* $1 + bVq \cdot \Omega$. The product of these four is, $Q = qA + r\Omega$, where A is a point of the axis, since $Vr \| Vq$. The above factors are denoted by T_1Q, T_2Q, U_1Q, U_2Q, respectively; also

$$TQ = T_1QT_2Q = \textit{augmentor of } Q = \textit{octonion tensor},$$

$$UQ = U_1Q \cdot U_2Q = \textit{twister of } Q = \textit{unit octonion}.$$

In terms of q, r, A, we have

$$TQ = Tq(1 + S \cdot r/q \cdot \Omega), \quad UQ = Uq \cdot A(1 + V \cdot r/q \cdot \Omega).$$

The above names are derived from the effects of these factors on a vector octonion or *motor*, whose axis intersects the axis of the multiplier at right angles. Any motor $\alpha A + \beta\Omega$ is characterized by a tensor ($T\alpha$), pitch ($S \cdot \beta/\alpha$), and axis ($V\beta/\alpha$ *from* A); of these Tq affects only $T\alpha$ (the tensor of a product = product of the tensors of the factors); $1 + a\Omega$ affects only the pitch, adding to it a, the pitch of the multiplier (pitch of product = sum of pitches of the factors); $1 + V \cdot r/q \cdot \Omega$ translates the axis of the motor the distance $V \cdot r/q$; $Uq \cdot A$ turns the axis of the motor round the axis of Q through the angle of q. Conversely, any octonion is the ratio of two motors, and in fact, octonion and motor stand in very much the relationship of quaternion and vector.

The symbols S, V, K have application as invariant operations, viz.:

$$SQ = Sq + Sr\Omega, \quad VQ = Vq \cdot O + Vr \cdot \Omega, \quad KQ = Kq \cdot O + Kr \cdot \Omega.$$

A complex scalar (SQ) is self-conjugate, and its square is a positive scalar (if the first term (Sq) determines the sign).

A complex vector, *i. e.*, a motor, is conjugate to its negative, and its square is a negative scalar. McAulay changes VQ to MQ because V does not connote with motor, but such change is unnecessary, and is not given weight in the equally important cases of T for augmentor, U for twister, s for convert, t for pitch, etc.

We have $Q = SQ + VQ$, $KQ = SQ - VQ$, $QKQ = KQ \cdot Q = (SQ)^2 - (VQ)^2 = (TQ)^2 \; K \cdot QQ' = KQ' \cdot KQ$, etc. If $QQ' = 0$, then either $Q = 0$ or $Q' = 0$ or $Q = r\Omega$, $Q' = r'\Omega$.

All results not depending upon divisions by a nil octonion $r\Omega$, are therefore identical in form with quaternion results, but with wider geometrical and physical meanings. For example, a motor $\alpha O + \beta\Omega = \alpha P + \beta'\Omega$, may be represented by the velocity system of a rigid body in which α is the vector angular velocity and β, β' the linear velocities at O, P. The axis is the instantaneous axis of rotation. This is what Ball calls a twist about a screw; the pitch of the motor is the pitch of the screw, which is right or left handed according as the pitch is positive or negative. If A be a point of the axis the motor takes the form $\alpha A + c\alpha\Omega$ where c is the pitch and $c\alpha$ the translation vector. The motor is also represented by a system of forces acting on a rigid body, which reduces to what Ball calls a wrench about a screw; α is the resultant force, and β, β' the moments of the system about O, P. It is also represented by a system of impulses, whose momentum is α, and moments of momentum about O, P are β, β'.

It will now be seen in what respect this analysis can prove valuable, and McAulay has not only developed this analysis very thoroughly in respect to linear functions of octonions, differentiations, nabla, and McAulayan differentiators, but he has applied it to develop and extend the results of Ball's theory of screws. It would seem that the "A process" which was presented by the reviewer before the Indiana academy of science in 1897, would prove a valuable addition to this theory. It is also possible that like quaternions (BULLETIN, November, 1897) the system may be extended to fourfold space with complete representation of the octonion.

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THESES IN MATHEMATICS AT THE UNIVERSITY OF PARIS.

1. *Sur quelques points de la théorie des fonctions*, par M. L. DESAINT. Paris, Gauthier-Villars, 1897. 4to, 74 pp.
2. *Sur une classe particulière de groupes hyperabéliens*, par M. HENRY BOURGET. Ibid., 1898. 4to, 90 pp.
3. *Sur l'intégration des équations de la chaleur*, par M. ÉDOUARD LEROY. Ibid., 1898. 4to, 258 pp.
4. *Les équations différentielles linéaires et la théorie des groupes*, par M. F. MAROTTE. Ibid., 1898. 4to, 92 pp.
5. *Essai sur une théorie générale de l'intégration et sur la classification des transcendantes*, par M. JULES DRACH. Ibid., 1898. 4to, 144 pp.*

THESE five theses were presented to the faculty of sciences of the University of Paris for the degree of doctor in mathematical science; they were sustained in order of time as placed above during the academic year from November, 1897, to June, 1898, inclusive.

Before proceeding to a brief review of each it may be interesting to observe that the creation of the faculties of mathematical and physical science in France dates from March 17, 1808.† The first degree of this kind conferred by the faculty of sciences at Paris was given to Pierre-Louise-Marie Bourdon in 1811; his two theses maintained respectively on the 19th and 23d of March of that year were 1° the motion of a solid body solicited by any accelerative forces and turning about a fixed point, and 2° the theory of the elliptic movement of the planets according to the principle of universal gravitation and its application to the determination of the masses of some planets.

From 1811 to 1890 inclusive there were conferred one hundred and eighty-four such degrees at Paris, and forty-four by the other universities of France.‡ The average age of the candidates is thirty years and a fraction; the majority lie in the range from twenty-three to thirty-four years, the average for each of these years being twelve; there were thirteen at twenty-three, eighteen (the maximum) at twenty-

* The memoirs of Desaint, LeRoy, and Drach are also published in the *Annales de l'Ec. norm. sup.*, for the respective years 1897, 1897-8, 1898; those of Bourget and Marotte appeared in the *Annales de la Faculté des Sciences de Toulouse* for 1898.

† See Maire's Catalogue des thèses de sciences soutenues in France de 1810 à 1890 inclusivement, Paris, H. Welter, 1892.

‡ For the same period the respective numbers in the physical sciences were 231 and 61; in the natural sciences, 236 and 67.

eight, and ten at thirty-four; the ages range from seventeen years to fifty-five; Joseph Bertrand was the youngest candidate; the next age is twenty, and with but four exceptions, every age from twenty to fifty-five is represented; five candidates attained the degree at twenty-one and it was conferred on five who had passed the fifty-year mark; the average age of the first ten of this period (1810–1890) was twenty-six years, that of the last ten is also twenty-six, the fraction being slightly larger for the former.

1. In the first part of his memoir Desaint studies the distribution of the zeros of a function. This problem is one way of approaching the general problem of the resolution of equations; he does not propose to solve the very difficult problem of determining their roots point by point, but to limit the regions of the plane of the complex variable within which a function can vanish. The case of a polynomial with real roots is one of the rare examples in which the zeros of a function can be found point by point; if certain roots of a polynomial are commensurable these are obtained immediately; as to incommensurable roots the author proceeds thus: the value of the root sought is placed between two commensurable numbers, one, a, greater, and the other, b, less; consider the segment ab of the axis of reals and to each zero make correspond an analogous segment; the ensemble E of these segments represents the precision with which the equation has been resolved; from the author's point of view E limits the region of the plane where the roots of the polynomial are found.

The first chapter presents the geometric method of which Desaint makes use; it rests upon the following simple principle: consider an ensemble of segments F starting from a point z; if all these segments are situated below a straight line D, their resultant is essentially different from zero and situated below D. Applying this to rational fractions the author is led to a fundamental theorem which he applies to the roots of a polynomial, to algebraic functions, and to the zeros of uniform functions with polar discontinuities. Certain propositions found complete the investigations of Lucas, Berloty, and Cesàro.

Desaint is occupied in the second chapter with definite integrals studied by Hermite. Theorems on uniform functions are arrived at relative to the distribution of the values of the variable which make these functions assume a given value u. A theorem is presented which is applied to elliptic and hyperelliptic integrals, and especially to hypergeometric integrals.

In the second part of the memoir the theorem of Cauchy is used to turn the preceding theorems to a solution of the problem of the distribution of the values of the variable which assign a given value to a uniform function. The problem of finding the value of a function at any point when its values on a contour are given has received a great deal of attention; the author proposes the inverse problem of finding the values of the variable which assign the value u to a function, when the function is given by its values along a contour. He calls particular attention to this proposition: Let $F(z)$ be a uniform function for which infinity is an ordinary point; trace a circle C enclosing all the discontinuities of $F(z)$ and as near as we wish to the convex contour of a minimal surface enclosing these points; designate by M the maximum modulus of $F(z)$ upon the circle C of radius R; the values of z for which $F(z)$ takes the value u are on the interior of the circle T concentric with C and having as radius $R\sqrt{2}\left(1+\frac{M}{|A-u|}\right)$. From this theorem we deduce with facility a general theorem on the continuity of functions. The second part closes with a study of integral functions and of integrals of differential equations.

M. Desaint's second thesis consisted of general theorems of Galois upon algebraic equations; application to resolution by radicals of equations of first degree and to the construction of regular polygons.

2. M. Bourget, at the suggestion of Professor Picard, studies the particular hyperabelian group pointed out by the latter geometer in his memoir in the first volume of the fourth series of *Liouville's Journal* and previously in the *Comptes Rendus* for 17 March, 1884.

This particular group has a double origin. We can derive it from the transformation of the first order of the abelian functions of the second kind;* or it can be considered as isomorphic to the group of transformations similar arithmetically of the quaternary form

$$u_1^2 - Du_2^2 + u_3u_4.$$

The thesis falls into three parts. In the first of these, after generalities on similar substitutions of quaternary quadratic forms and on the arithmetic study of such forms, Bourget shows how the group which he studies is situated relative to the analogous groups considered by Goursat and Bianchi.

* Hermite, "Sur la transformation des fonctions abéliennes," *Comptes rendus*, vol. 40 (1855).

The second part is devoted to the study of the group. Seeking first the explicit form of the substitutions, the discontinuity for the imaginary values $\xi = \xi_1 + i\xi_2$, $\eta = \eta_1 + i\eta_2$ appertaining to the domain δ ($\xi_2 > 0$, $\eta_2 > 0$) is then demonstrated; the substitutions of the group are reduced to five of them, which fundamental substitutions present a remarkable analogy to the fundamental substitutions of the modular group; the author also brings to light an infinite number of subgroups analogous to the congruence subgroups of the modular group.

Bourget inquires in the third part how the ten functions

$$\vartheta(0,0/\tau_{11}, \tau_{12}, \tau_{22})$$

even with zero arguments, are affected by the transformations of the group, and from this study deduces a process which, by the aid of the theta functions, constructs an infinitude of functions which are reproduced by all the substitutions of the group. The moduli of Borchardt and Richelot are invariant under certain subgroups. In conclusion the general properties, according to Picard, of the fonctions of the group are given, and it is shown that, notwithstanding the cusp situated on the limit of the domain δ (whose existence was established in the second part), any three of these functions are connected by an algebraic relation.

M. Bourget's second thesis was the exposition of the method of Gauss for the calculation of secular variations.

3. The object of LeRoy's memoir is the resolution of certain problems of the integral calculus which support the mathematical theory of heat. These problems are referred to certain partial differential equations, of which Laplace's equation is the type. Each of them consists in establishing a principle analogous to that of Dirichlet.

The author proposes 1° to construct an integral $V(x, y, z)$ of the equation

$$\Delta V + aV_x + bV_y + cV_z = f(x, y, z, V) + \varphi(x, y, z),$$

which shall be definite and continuous in every point of a closed domain, on the boundary of which the function sought will be constrained to take values given in advance;

2° to design an integral $V(x, y, z, t)$ of the more general equation which he calls the equation of Fourier

$$\Delta V + a(x, y, z)V_x + b(x, y, z)V_y + c(x, y, z)V_z = f(x, y, z, V) + \varphi(x, y, z, t) + \psi(x, y, z)V_t,$$

where $adx + bdy + cdz$ is an exact integral, which integral

shall be definite and continuous for every point of a closed domain and for every positive value of the time, reducing for $t = 0$ to a function (x, y, z) given in advance and taking at the surface of the body values assigned *a priori*.

The plan followed in the paper may be briefly sketched thus: A remark made by M. Paraf* permits the author to extend the principal propositions relative to Laplace's equation to equations of a general type whatever be the number of variables; then having reduced the method called by Poincaré the méthode du balayage to a canonical form, he deduces by a uniform process the complete resolution of the problems of the theory of heat, at least so far as existence theorems are concerned.

The equations of thermic equilibrium are considered from the point of view of the generalization of Dirichlet's principle; a point of view taken by Picard in several well-known memoirs.† The calculations are made in three point variables. The méthode du balayage employed is independent of any preliminary theory of Laplace's equation; for linear equations the author demonstrates the principles of Dirichlet and its generalization at one stroke applicable alike to plane and space; as to non-linear equations a new method is proposed, which succeeds for all such equations as are suggested by the theory of heat.

In the second part the author studies the properties of functions which he calls the fundamental harmonic functions attached to a closed surface; his guide here is a memoir of Poincaré.‡ The existence theorems demonstrated lead to developments in series by which, after Lamé, the construction of the solution of Dirichlet's problem would be sought. The author is limited here to Laplace's equation.

He arrives finally at the equations of cooling of solid bodies and to Fourier's problem. For the case of a homogeneous body an imitation of the méthode du balayage is employed. The author shows that it is again possible to find, *a posteriori*, solutions under the form of series to which he gives the name of Lamé; the process throws new light on several questions of physics; for example, the problem of vibrating membranes. In conclusion LeRoy considers the possibility of applying to the general equations of a variable régime the process imagined by Picard for the equations of permanent régime.

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* A. Paraf, "Sur le problème de Dirichlet et son extension au cas de l'équation linéaire générale," *Toulouse Annals*, vol. 6 (1892).

† Picard, *Acta Math.*, vol. 12; *Liouville's Journal*, 1890, 1896; *Jour. de l'Éc. poly.*, cahier 60.

‡ Poincaré, *Rend. del circ. mat. di Palermo*, 1894; *Acta Math.*, 1896.

M. LeRoy's second thesis was on the integrals of total differentials and on the double integrals of the first kind in the theory of algebraic surfaces.

4. Marotte's memoir has to do with the question of extending to linear differential equations the ideas introduced by Galois into the theory of algebraic equations. The point of view was discovered by Picard* in the following theorem: To every nth order linear differential equation there corresponds an algebraic group of linear transformations in n variables, which possesses properties analogous to those of the Galois group of an algebraic equation.

Vessiot† demonstrated completely the double property of this group of rationality and made evident its close relation to the problem of integration. Drach‡ extended the theory of Galois to partial differential equations of the first order.

The first part of Marotte's paper is occupied with an analytical study of the singularities of linear differential equations having rational coefficients and the classification of the transcendents which integrate them. The author shows that to each singular point a of such an equation there is attached a group of linear transformations which plays the same rôle in the study of a singularity as does the group of Galois in the resolution of an algebraic equation, or the group of rationality in the integration of a linear differential equation. This group is called the group of meromorphism, because its differential invariants are expressed by functions of x meromorphic at point a; the group characterizes the nature of the singular point. For an nth order equation there are as many classes of singular points as there are subgroups in the linear group of n variables.

After having established the relations which exist between the group of rationality, the group of monodromism and the groups of meromorphism, the author enlarges the field of application of the methods of Galois and shows how they lead to the notion of group of monodromism. The group of rationality gives the position of the integrals with respect to the ensemble of rational functions exactly as the group of monodromism gives their position relative to the ensemble of uniform functions. The preceding results are then applied to the classification of the transcendents which integrate linear differential equations with rational coefficients; the notion species of Riemann is the ultimate classifying element. A method is presented by which it can be recognized when two given equations are of the same species.

* *Comptes rendus*, 1883; *Toulouse Annals*, 1887.
† *Ann. de l'Ec. norm.*, 1892.
‡ *Comptes rendus*, 1893, 1895.

The second part of the memoir is devoted to the determination of the group of rationality of a differential equation with rational (or algebraic) coefficients. A new classification of homogeneous linear groups enables the author to refer the determination of the group of rationality to the resolution of the problem of determining whether a linear differential equation with rational coefficients admits of an integral whose logarithmic derivative is rational or algebraic; hence the conclusion that we can always determine the group of rationality of a linear equation of order two, three or four, or refer the determination to the study of an abelian integral. The author thus finds all the possible cases of reduction of a linear equation of the fourth order. He refers the integration problem to its canonic form. The method used extends itself immediately to equations of higher order.

In the concluding chapter the work of von Koch* is applied to the study of the question of finding whether a linear equation with rational coefficients admits of an integral whose logarithmic derivative is meromorphic (normal integral), to which problem Marotte refers the determination of the group of monodromism attached to a singular point of a linear equation.

M. Marotte's second thesis was on the general principles of dynamics.

5. The remarkable thesis of Drach calls for more extended notice than can be given to it in the space at command for this article. A suitable review of the memoir will appear in a subsequent number of the BULLETIN.

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NOTES.

AT the annual meeting of the Society, December 28, 1899, President R. S. WOODWARD will deliver a presidential address on "The century's progress in applied mathematics."

THE president (Lord KELVIN), the vice-presidents and the secretaries of the London mathematical society have been renominated to serve in the same capacity on the council for the ensuing year. Professor W. BURNSIDE,

* *Acta Math.*, vol. 18; *Comptes rendus*, 1893.

Mr. H. M. MACDONALD and Mr. E. T. WHITTAKER, have been nominated to fill the vacancies on the council. At the annual meeting which was held on November 9th, the DeMorgan medal was presented to the sixth medalist, Professor BURNSIDE. The Council have sanctioned the issue by the secretaries of an index to the first thirty volumes of the *Proceedings*, which will be drawn up on the same lines as that of the first fifty volumes of the *Mathematische Annalen*. They have also authorized Mr. R. TUCKER to draw up a list of all the members elected since the foundation of the society in 1865.

THE American physical society held its first regular meeting at Columbia University, October 28th. It is proposed to hold four meetings annually, simultaneously with the AMERICAN MATHEMATICAL SOCIETY. The present officers are Professors H. A. ROWLAND, president; A. A. MICHELSON, vice-president; E. MERRITT, secretary; W. HALLOCK, treasurer.

THE recently organized Astronomical and astrophysical society of America has elected the following officers: president, Professor SIMON NEWCOMB, of Johns Hopkins University; first vice-president, Professor CHARLES A. YOUNG, of Princeton University; second vice-president, Professor GEORGE E. HALE, of Yerkes Observatory; secretary, Professor GEORGE C. COMSTOCK, of Washburn Observatory; treasurer, Professor C. L. DOOLITTLE, of the University of Pennsylvania.

The Festschrift prepared in honor of the seventieth birthday, August 23, 1899, of Professor MORITZ CANTOR, of Heidelberg, the well-known historian of mathematical science, has just appeared as a supplement to the forty-fourth volume of the *Zeitschrift für Mathematik und Physik*, from the press of Teubner, under the editorship of Professors M. CURTZE, of Thorn, and S. GÜNTHER, of Munich. The volume of more than six hundred pages contains a heliogravure portrait of Professor Cantor, a catalogue of his scientific work, and the following thirty-two memoirs devoted to historical subjects:

"Développement des procédés servants à décomposer le quotient en quantièmes," by V. V. BOBYNEN, of Moscow; "Zur Geschichte der prosthaphaeretischen Methode in der Trigonometrie," by A. v. BRAUNMÜHL, of Munich; "Notes on the history of logarithms," by FLORIAN CAJORI, of Colorado Springs; "Der Tractatus Quadrantis des Robertus

Anglicus in deutscher Uebersetzung aus dem Jahre 1477," by M. CURTZE, of Thorn; "Zur Geschichte der Prinzipien der Infinitesimalrechnung. Die Kritiker der 'Théorie des fonctions analytiques' von Lagrange," by S. DICKSTEIN, of Warsaw; "P. W. Wargentin und die sogenannte Halley'sche Methode. Ein Beitrag zur Geschichte der mathematischen Statistik," by G. ENESTRÖM, of Stockholm; "Intorno ad un inedito e sconosciuto Trattato di Mechaniche di Galileo-Galilei nell'Archivo di S. A. il Principe di Thurn-Taxis in Ratisbona," by A. FAVARO, of Padua; "Zur Geschichte der Längenbestimmung zur See," by E. GELCICH, of Triest; "Die Geometrie von Le Clerc und Ozanam, ein interessantes mathematisches Plagiat aus dem Ende des XVII. Jahrhunderts," by J. H. GRAF, of Bern; "Nikolaus von Cusa und seine Beziehungen zur mathematischen und physikalischen Geographie," by S. GÜNTHER, of Munich; "On an allusion in Aristotle to a construction for parallels," by T. S. HEATH, of Cambridge, England; "Byzantinische Analekten," by J. L. HEIBERG, of Copenhagen; "Ueber die Aufgaben einer Geschichte der Physik," by A. HELLER, of Budapest; "Winkelmessungen durch die Hipparchische Dioptra," by F. HULTSCH, of Dresden; "Des Rheticus Canon doctrinæ triangulorum und Vieta's Canon mathematicus," by K. HUNRATH, of Rendsburg; "Il 'Giornale de Letterati d'Italia' di Venezia e la 'Raccotta Calogerà' come fonti per la storia delle matematiche nel secolo XVIII.," by G. LORIA, of Genoa; "Notes sur le caractère géométrique de l'ancienne astronomie," by P. MANSION, of Gand; "Ueber die Encyclopädie der mathematischen Wissenchaften," by W. F. MEYER, of Königsberg; "Zur Terminologie der ältesten mathematischen Schriften in deutscher Sprache," by F. MÜLLER, of Loschwitz; "Die Rechenmethoden auf dem griechischen Abakus," by A. NAGL, of Vienna; "Die Geschichte der exakten Wissenschaften und der Nutzen ihres Studiums," by F. ROSENBERGER, of Frankfort; "Die Unverzagt'schen Linienkoordinaten. Ein Beitrag zur Geschichte der analytischen Geometrie," by F. RUDIO, of Zurich; "Franz Adolph Taurinus. Ein Beitrag zur Vorgeschichte der nichteuklidischen Geometrie," by PAUL STÄCKEL, of Kiel; Johann Scheubel, ein deutscher Algebraiker des XVI. Jahrhunderts, by H. STAIGMÜLLER, of Stuttgart; "Mathematik bei den Juden (1501–1550)," by M. STEINSCHNEIDER, of Berlin; "Bemerkungen zur Geschichte der altgriechischen Mathematik," by A. STURM, of Seitenstetten; "Der Loculus Archimedius oder das Syntemachion des Arch-

imedes. Zum ersten Male nach zwei Manuscripten der Kgl. Bibliothek zu Berlin herausgegeben und übersetzt," by H. SUTER, of Zurich; "Les 'Excerpta ex M. SS. R. Des-Cartes,'" by PAUL TANNERY, of Pantin; "Einige Additionsmaschinen," by F. A. UNGER, of Leipzig; "Zur Geschichte der deutschen Algebra," by E. WAPPLER, of Zwickau; "Pierre Fermat's Streit mit John Wallis. Ein Beitrag zur Geschichte der Zahlentheorie," by G. WERTHEIM, of Frankfort; "Die Entdeckung der Parabelform der Wurflinie," by E. WOHLWILL, of Hamburg.

THE following recently issued catalogues of new and second hand scientific publications are more or less rich in mathematical works:—catalogue 40, Karl Krebs, Giessen, Germany; catalogue 276, Macmillan and Bowes, Cambridge, England; catalogue 173, Mayer and Müller, Berlin, Germany; catalogue 92, Oswald Weigel, Leipsic, Germany. M. A. Hermann, of Paris, will issue a new catalogue this month (November).

OXFORD UNIVERSITY. The courses of lectures announced for the fall term of 1899 include the following on mathematical subjects:—By Professor W. ESSON: Analytic theory of place curves; Synthetic theory of place curves.—By Professor H. TURNER: Mathematical astronomy.—By Professor A. E. H. LOVE: Electricity and magnetism.—By Professor E. ELLIOTT: Theory of numbers.

CAMBRIDGE UNIVERSITY. Mathematical courses are announced as follows for the current academic year: *Michaelmas term*, 1899:—By Professor G. G. STOKES: Hydrodynamics, three hours.—By Professor A. R. FORSYTH: Theory of differential equations, three hours; Calculus of variations, two hours.—By Professor G. H. DARWIN: Lunar theory, three hours.—By Mr. R. PENDLEBURY: Theory of equations, three hours; Spherical astronomy, three hours.—By Dr. E. W. HOBSON: Spherical and cylindrical harmonics, three hours.—By Mr. J. LARMOR: Electricity and magnetism, three hours.—By Mr. H. F. BAKER: Theory of functions, three hours.—By Mr. H. M. MACDONALD: Waves (especially waves of light), three hours.—By Mr. H. W. RICHMOND: Plane analytical geometry, three hours.—By Mr. G. T. WALKER: The electromagnetic field, three hours.—By Mr. E. T. WHITTAKER: The problem of three bodies, three hours.—By Mr. J. H. GRACE: Invariants and geometrical applications, three hours.—*Lent term*, 1900:—By Professor G. G. STOKES: Physical optics.

—By Professor A. R. FORSYTH: The differential equations of theoretical dynamics.—By Professor R. S. BALL: Planetary theory.—By Mr. R. PENDLEBURY: Theory of numbers.—By Dr. E. W. HOBSON: Vibrations and sound.—By Mr. J. LARMOR: Electricity and magnetism (continuation).—By Mr. H. F. BAKER: Theory of functions (continuation).—By Mr. H. M. MACDONALD: Elasticity.—By Dr. J. W. L. GLAISHER: Elliptic functions.—By Mr. R. A. HERMAN: Hydrodynamics.—By Mr. A. N. WHITEHEAD: Universal algebra.—By Mr. A. BERRY: Elliptic functions.—By Mr. G. T. WALKER: Physical optics.—By Mr. G. T. BENNETT: Linear and quadratic complexes.—*Easter term*, 1900:—By Mr. W. L. MOLLISON: Theory of potential and electrostatics.—By Mr. E. T. WHITTAKER: Analysis.—*Long vacation*, 1900:—By Professor G. H. DARWIN: Attraction and potential.

During the winter semester 1899–1900 the several universities mentioned below offer the following mathematical courses:

UNIVERSITY OF GIESSEN. By Professor M. PASCH: Functions of complex variables, four hours; Seminar, two hours.—By Professor E. NETTO: Analytical mechanics, four hours; Seminar, two hours.—By Professor R. HAUSSNER: Differential and integral calculus, four hours; Perspective, two hours; Theory of probabilities, one hour; Exercises in the elements of higher mathematics, one hour.

UNIVERSITY OF MARBURG. By Professor F. SCHOTTKY: Integral calculus, four hours; General theory of functions, five hours; Seminar, two hours.—By Professor A. E. HESS: Geometry of space, synthetic and analytic, four hours; Theory of algebraic equations and determinants, four hours; Seminar, two hours.—By Dr. F. DALWIGK: Introduction to higher mathematics and exercises, for hours; Selected subjects of the theory of functions, two hours; Exercises in the construction of mathematical models, one hour.

UNIVERSITY OF PRAGUE. By Professor F. Lippich: Theoretical mechanics, three hours; Theory of capillarity, two hours; Seminar, two hours.—By Professor G. PICK: Differential and integral calculus, three hours; Algebraic analysis, two hours; Seminar, two hours.—By Professor K. BOBEK: Curves and surfaces in space, two hours; Descriptive geometry and constructions, four hours.

UNIVERSITY OF STRASSBURG. By Professor T. REYE: Analytical geometry of space (new methods), three hours; Mathematical theory of elasticity of solid bodies, two hours; Seminar, two hours.—By Professor H. WEBER: Integration of the partial differential equations of mathematical physics, four hours; Introduction to higher mathematics, two hours; Exercises in mathematical seminar, three hours.—By Professor G. ROTH: Algebraic analysis and determinants, three hours; Analytical geometry of space, two hours; Ordinary differential equations, two hours.—By Professor A. KRAZER: Differential and integral calculus, four hours; Analytical geometry of the plane, three hours; Exercises in infinitesimal calculus, two hours.—By Dr. E. TIMERDING: Vector analysis, two hours.—By Dr. J. WELLSTEIN: Descriptive geometry and exercises, four hours; Seminar in coöperation with Professor WEBER, two hours.

UNIVERSITY OF TÜBINGEN. By Professor A. VON BRILL: Introduction to higher mathematics, four hours; Theory of algebraic curves, three hours; Seminar, two hours.—By Professor H. STAHL: Higher analysis, three hours; Partial differential equations, three hours; Seminar, two hours.—By Professor L. MAURER: Descriptive geometry and exercises, three hours; Elements of differential and integral calculus, two hours; Exercises in algebra and geometry for students of natural science, two hours.

UNIVERSITY OF VIENNA. By Professor G. VON ESCHERICH: Elements of differential and integral calculus, five hours; Exercises on the preceding course, two hours; Proseminar, one hour; Seminar, two hours.—By Professor L. GEGENBAUER: Algebra, four hours; Spherical and cylindrical harmonics, one hour; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: On the laws of reciprocity, five hours; Seminar, two hours; Proseminar, one hour.—By Professor G. KOHN: Analytical geometry and exercises, four hours.—By Dr. V. SERSAWY: The mathematics of life insurance, two parts, of three and four hours, respectively.—By Dr. A. TAUBER: Insurance mathematics and exercises, six hours.—By Dr. K. ZINDLER: Differential equations, two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics, three hours.

UNIVERSITY OF WURZBURG. By Professor F. PRYM: Differential calculus, four hours; Higher theory of functions, four hours; Seminar in differential calculus, two hours; Seminar in selected sections of higher mathematics,

two hours.—By Professor A. Voss: Analytical mechanics, four hours; Exercises in analytical mechanics, two hours; Algebra, three hours; Seminar in synthetic and analytic geometry, one hour.

UNIVERSITY OF ZURICH. By Professor H. BURKHARDT: Differential and integral calculus, four hours; mechanics, four hours; Seminar, two hours.—By Dr. A. WEILER: Descriptive geometry and exercises, four hours; Analytical geometry and exercises, four hours.—By Dr. F. KRAFT: Differential equations, three hours; Analytical geometry, three hours; Theory of probabilities, two hours; Seminar, two hours.—By Dr. E. GUBLER, algebra, three hours; spherical trigonometry, one hour; methods of mathematical instruction, two hours.

ANDREW GRAY, F.R.S., professor of physics in the University of North Wales since 1884, has been appointed professor of natural philosophy in Glasgow University, to succeed Lord KELVIN, who recently resigned.

A STATUE erected in memory of the late Professor F. TISSERAND was unveiled at Nuits-Saint-Georges on the 15th of October.

MR. LINDSAY DUNCAN has been appointed instructor in mathematics at Union College, and MR. EDWIN HAVILAND has been made assistant in mathematics in Swarthmore College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BURKHARDT (H.). See ENCYKLOPÄDIE.

BURNSIDE (W. S.) and PANTON (A. W.). Introduction to determinants; being a chapter from the theory of equations. London, Longmans, 1899. 8vo. 2s. 6d.

——. Theory of equations. 4th edition. Vol. I. London, Longmans, 1899. 8vo. 9s. 6d.

CAPELLI (A.). Lezioni di algebra complementare, ad uso degli aspiranti alla licenza universitaria in scienze fisiche e matematiche. 2a edizione con aggiunte. Napoli, Pellerano, 1898. 8vo. 16 + 680 pp. Fr. 8.50

CORDARA (S.). Nota sulla impossibilità della quadratura del cerchio. Bologna, Azzoguidi, 1899. 8vo. 11 pp.

DETER (J.). Repetitorium der Differential- und Integralrechnung. 3te Auflage. Berlin, Rockenstein, 1899. 8vo. 119 pp. M. 1.60

DICKSTEIN (S.). See KLEIN (F.).

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Mit Unterstützung der Akademien der Wissenschaften zu München und Wien und der Gesellschaft der Wissenschaften zu Göttingen, sowie unter Mitwirkung zahlreicher Fachgenossen. [In 2 Teilen.] Teil I. Reine Mathematik, herausgegeben von H. Burkhardt und W. F. Meyer Vol. I. Arithmetik und Algebra, redigiert von W. F. Meyer. Heft 3, enthaltend: E. Netto, Rationale Funktionen einer Veränderlichen; ihre Nullstellen.—E. Netto, Rationale Funktionen mehrerer Veränderlichen.—G. Landsberg, Algebraische Gebilde; arithmetische Theorie algebraischer Grössen.—W. F. Meyer, Invariantentheorie.—Leipzig, Teubner, 1899. 8vo. Pp. 225–352. M. 3.80

GRAHAM (J.). Elementary treatise on practical mathematics for technical colleges and schools. London, Arnold, 1899. 8vo. 284 pp. (Mathematical series.) 3s. 6d.

GULDBERG (A.). Sur la théorie des solutions singulières des équations aux différentielles totales du premier ordre. Christiania, Dybwad, 1899. 8vo. 26 pp. (*Videnskabsselskabets Skrifter*, I. Math.-naturv. Klasse, 1899, No. 4).

HESSENBERG (G.). Ueber die Invarianten linearer und quadratischer binärer Differentialformen und ihre Anwendung auf die Deformation der Flächen. Berlin, 1899. 4to. 50 pp. M. 2.50

HUYGENS (C.). Oeuvres complètes, publiées par la Société hollandaise des sciences. Vol. VIII: Correspondance 1676–84. La Haye. 1899. 4to. 631 pp., 7 plates, 1 portrait. Fr. 30.00

KLEIN (F.). Odczyty o matematyce miane w Evanston od 28 sierpnia do 9 wrzesnia 1893 r. dla członków kongresu matematycznego odbytego w czasie wystawy wszechświatowej w Chicago. Spisane przez A. Ziweta, Przełożył za upoważnieniem autora S. Dickstein. Warszawa, 1899. 8vo. 6 + 111 pp., with Klein's portrait.

KRAUSS (E.). See PINET (H.).

LANDAU (E.). Neuer Beweis der Gleichung $\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$. Berlin, 1899. 8vo. 16 pp. M. 1.00

LANDSBERG (G.). See ENCYKLOPÄDIE.

LINDELÖF (E.). Remarques sur un principe général de la théorie des fonctions analytiques. Helsingfors, 1898. 4to. 39 pp. M. 2.00

——. Recherches sur les polyèdres maxima. Helsingfors, 1898. 4to. 47 pp. M. 2.00

MEYER (W. F.). See ENCYKLOPÄDIE.

MILLER (G. A.). Memoir on the substitution-groups whose degree does not exceed eight. 4to. (*American Journal of Mathematics*, Vol. 21, pp. 287–338.)

MÜLLER (H.). Die Lehre von den Coordinaten und Kegelschnitten, für den Unterricht dargestellt. Berlin, Moeser, 1899. 8vo. [From

Müller, Die Mathematik auf den Gymnasien und Realschulen.] Pp. 165–216. Boards. M. 1.00

NETTO (E.). See ENCYKLOPÄDIE.

PANTON (A. W.). See BURNSIDE (W. S.).

PASCAL (E.). Repertorium der höheren Mathematik (Definitionen, Formeln, Theoreme, Literaturnachweise). Autorisierte deutsche Ausgabe von A. Schepp. (In 2 Teilen.) Teil I: Die Analysis. Leipzig, Teubner, 1899. 8vo. Cloth.

PINET (H.). Mémoire sur une nouvelle méthode pour la résolution des équations numériques. Suivi d'un appendice donnant le détail des opérations, par E. Krauss. Paris, Nony, 1899. 4to. Pp. 47.

RAZZABONI (A.). Le formole di Frenet in geometria iperbolica e loro principali applicazioni. Bologna, Gamberini e Parmeggiani, 1899. 4to. 22 pp. Fr. 2.50

SCHEPP (A.). See PASCAL (E.).

SEVERINI (C.). Sull'integrazione approssimata delle equazioni differenziali ordinarie. Bologna, Zanichelli, 1899. 4to. 27 pp. Fr. 3.00

ZIGNAGO (I.). Osservazioni sulla convergenza delle serie. Genova, Instituto Sordomuti, 1899. 16mo. 10 pp.

ZIWET (A.). See KLEIN (F.).

II. ELEMENTARY MATHEMATICS.

BELLACCHI (G.). Lezioni ed esercizî di algebra complementare. Fascicolo 2. Firenze, Barbèra, 1899. 16mo. 143 pp. Fr. 2.00

DETER (J.). Mathematisches Formelbuch für höhere Unterrichtsanstalten. Neu herausgegeben von E. Arndt. 4te Auflage. Berlin, Rockenstein, 1899. 8vo. 5 + 58 pp. M. 0.90

FAIFOFER (A.). Trattato di geometria intuitiva, ad uso delle scuole tecniche e normali. 31a edizione. Venezia, Tipografia Emiliana, 1899. 16mo. 165 pp. Fr. 2.00

——. Elementi di trigonometria piana e tavole logaritmico-trigonometriche ad uso dei licei. 11a edizione. Venezia, Tipografia Emiliana, 1899. 16mo. 92 + 69 pp. Fr. 2.00.

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 58ste Auflage. Halle, Strien, 1899. 8vo. 166 + 35 pp. Half leather. M. 2.50

——. Vierstellige logarithmisch-trigonometrische Handtafel. Stereotyp-Druck. 3te Auflage. Halle, Strien, 1899. Folio. M. 0.60

——. Dieselbe, für Decimalteilung des Quadranten. Stereotyp-Druck. 2te Auflage. Halle, Strien, 1899. Folio. M. 0.80

GOSETTI (L.). Corso di aritmetica e di algebra. Parte I. Venezia, Ferrari, 1899. 8vo. 201 pp. Fr. 2.00

HUC (E.). Algèbre, à l'usage des candidats à la première partie des baccalauréats. Bar-le-Duc, Comte-Jacquet, 1899. 16mo. 104 pp.

LEONARDI (O.). Elementi di algebra. Foligno, Salvati, 1899. 16mo. 47 pp.

MACKAY (J. S.). Arithmetic theoretical and practical. London and Edinburgh, W. and R. Chambers, 1899. 8vo. 11 + 472 pp.

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. (In 2 Teilen.) Teil I: Die Unterstufe: Lehraufgabe der Klassen Quarta bis Unter-Sekunda. 8 + 152 pp.—Teil II: Die Oberstufe: Lehraufgabe der Klassen Ober-Sekunda und Prima. 10 + 216 pp. Berlin, Moeser, 1899. 8vo. Cloth. M. 5.70

NEOVIUS (L.). Elementarkurs i algebra. Del I: Rationella expressioner. Helsingfors, 1898. 8vo. 10 + 111 pp. M. 2.00

——. Lärobok i plan trigonometri. Helsingfors, 1899. 8vo. 10 + 119 pp. M. 2.00

PADOA (A.). Note critiche agli Elementi di geometria, di G. Veronese. Pinerolo, Chiantore-Mascarelli, 1899. 8vo. 22 pp.

——. Note critiche al Libro di aritmetica e di algebra elementare, di P. Cazzaniga. Pinerolo, Chiantore-Mascarelli, 1899. 8vo. 17 pp.

PUCHETTI (N.). Aritmetica e geometria per la quarta classe elementare. Torino, Paravia, 1899. 16mo. 78 pp. Fr. 0.60

——. Aritmetica e geometria per la quinta classe elementare. Torino, Paravia, 1899. 16mo. 84 pp. Fr. 0.60

RIDDEL (J.). Practical plane and solid geometry. With exercises and examination questions. Edinburgh, Oliver & Boyd, 1899. 8vo. 336 pp. 2s.

SCHLÖMILCH (O.). Fünfstellige logarithmische und trigonometrische Tafeln. Wohlfeile Schulausgabe. 15te Auflage. Braunschweig, 1899. 8vo. 151 pp. M. 1.00

SCHUSTER (M.). Geometrische Aufgaben. Ein Lehr- und Uebungsbuch zum Gebrauch beim Unterricht an höheren Schulen. Ausgabe A: für Vollanstalten. Leipzig, Teubner, 1899. 8vo. 8 + 147 pp., 2 plates. Cloth. M. 2.00

STEREOSCOPIC VIEWS of solid geometry figures; with references to Wells' "Essentials of solid geometry." Boston, Heath, 1899. 93 cards in a pasteboard box. $0.60

VALLE (G.). Funzioni ad una variabile e loro limiti; un capitolo di algebra elementare. Noto, Zammit, 1899. 8vo. 38 pp.

WELLS (W.). New higher algebra. Boston, Heath, 1899. 12mo. 8 + 446 pp. Half-leather. $1.25

——. See STEREOSCOPIC VIEWS.

III. APPLIED MATHEMATICS.

BRYAN (G. H.) and ROSENBERG (F.). First stage mechanics of fluids. For elementary examination of Science and art department. 2d edition. London, Clive, 1899. 12mo. 224 pp. (Organic science series.) 2s.

CLERICI (V.). Considerazioni intorno alle formole per la misura delle acque correnti. Firenze, Ricci, 1899. 8vo. 18 pp.

CREW (H.). The elements of physics, for use in high schools. New York, The Macmillan Company, 1899. 12mo. 14 + 347 pp. $1.10

EDDY (H. T.). The graphical treatment of alternating currents in branching circuits. With special reference to the case of variable frequency. 8vo. (A paper to be presented at the fifteenth general meeting of the American Institute of electrical engineers, Omaha, June 30, 1898.) Pp. 511–554.

EDDY (H. T.), MORLEY (E. W.), and MILLER (D. C.). The velocity of light in the magnetic field. 8vo. (*Physical Review*, Vol. VII., December, 1898. Pp. 283–295.)

GUIDI (C.). Lezioni sulla scienza delle costruzioni. Parte II: Teoria dell' elasticità e resistenza dei materiali. 3a edizione. Torino, Bertolero, 1899. 8vo. 7 + 240 pp., 7 plates. Fr. 6.00

HUTTON (F. R.). Heat and heat engines; a study of the principles which underlie the mechanical engineering of a power plant. New York, Wiley, 1899. 8vo. 21 + 553 pp. Cloth. $5.00

JORDAN (W.). Hülfstafeln für Tachymetrie. 2te Auflage. Stuttgart, Metzler, 1899. 8vo. 15 + 246 pp. M. 8.00

MADAMET (A.). Tiroirs et distributeurs de vapeur; appareils de mise en marche et de changement de marche. 2me édition. Paris, Gauthier-Villars, 1899. 16mo. 151 pp. (Encyclopédie des aide-mémoire, section de l'ingénieur.)

MAZZOTTO (D.). Sugli armonici delle vibrazioni elettriche; nota. Pisa, Pieraccini, 1899. 8vo. 8 pp.

MERRIMAN (M.). Elements of precise surveying and geodesy. New York, Wiley, 1899. 8vo. 261 pp. Cloth. $2.50

MILLER (D. C.). See EDDY (H. T.).

MORLEY (E. W.). See EDDY (H. T.).

PORTUGAL DE FARIA (A. DE). See SOARES DE BARROS.

REINHERTZ (C.). Geodäsie. Einführung in die wesentlichsten Aufgaben der Erdmessung und Landesvermessung. Leipzig, 1899. 12mo. 181 pp. Cloth. M. 0.80

REUTER (J.). Geometriska uppgifter fran det praktiska. omrade Helsingfors, 1898. 4to. 65 pp. M. 2.00

REY-PAILHADE (J. DE). Extension du système métrique à la mesure du temps et des angles; conférence faite sous les auspices de la chambre syndicale d'horlogerie, le 23 juin 1898, à l'Ecole d'horlogerie de Paris. Montdidier, Carpentier, 1899. 8vo. 40 pp. 5 plates.

ROSENBERG (F.). See BRYAN (G. H.).

SAVANDER (O.). Détermination relative de la pesanteur à Helsingfors précédée d'un aperçu sur les formules de réduction. Helsingfors, 1898. 8vo. 3 + 195 pp. 2 plates. M. 4.00

SOARES DE BARROS e VASCONCELLOS (J. J.). Oeuvres, reproduites par A. de Portugal de Faria. I: Nouvelles équations pour la perfection de la théorie des satellites de Jupiter et pour la correction des longitudes terrestres déterminées par les observations des mêmes satellites.—II: Observations et explications de quelques phénomènes vus dans le passage de Mercure au devant du disque du soleil, observé à l'hotel de Cluny à Paris le 6 may 1753, et leur application pour la perfection de l'astronomie. Livourne, Giusti, 1899. 8vo. 73 pp.